

Elements of harmonic analysis

Stephen William Semmes
Rice University
Houston, Texas

Contents

1	Finite abelian groups	1
2	Riesz–Thorin convexity	5
3	Continuous functions on \mathbf{R}^n	13
4	Fourier transforms	20

These informal notes are based on a course given at Rice University in the spring semester of 2004, and much more information can be found in the references.

1 Finite abelian groups

Let A be a finite abelian group. Thus A is a nonempty set equipped with a binary operation which we denote $+$, which is to say that if a, b are elements of A , then $a + b$ is a well-defined element of A . This operation is commutative and associative, which is to say that

$$(1.1) \quad a + b = b + a$$

and

$$(1.2) \quad (a + b) + c = a + (b + c)$$

for all $a, b, c \in A$. There is an identity element 0 in A , which is characterized by the property

$$(1.3) \quad a + 0 = a$$

for all $a \in A$, and for each $a \in A$ there is a unique inverse element, denoted $-a$, which is characterized by

$$(1.4) \quad a + (-a) = 0.$$

By a *character* on A we mean a function ϕ from A to the complex numbers with modulus 1 such that

$$(1.5) \quad \phi(a + b) = \phi(a) \phi(b)$$

for all $a, b \in A$. In other words, ϕ is a homomorphism from A into the group \mathbf{T} of complex numbers with modulus 1 using multiplication as the group operation. Let us note that if ϕ is a homomorphism from A into the nonzero complex numbers using multiplication as the group operation, then ϕ automatically takes values in \mathbf{T} . Indeed, for each element a of A , there is a positive integer m so that the sum of a n times is equal to 0, and this leads to the conclusion that $\phi(a)^m = 1$. In other words, $\phi(a)$ is an m th root of unity, and it has modulus equal to 1 in particular.

Let A^* denote the set of characters on A . Note that we automatically have a *unit character* on A , which sends every element of A to 1. Also, if ϕ_1, ϕ_2 are characters on A , then so is the product $\phi_1 \phi_2$, and if ϕ is a character on A , then $1/\phi = \bar{\phi}$ is a character on A . Here we use the standard notation that if $z = x + iy$ is a complex number with x, y real numbers, the real and imaginary parts of z , then \bar{z} is the *complex conjugate* of z , given by $\bar{z} = x - iy$. In short A^* becomes an abelian group with respect to multiplication of characters, called the *dual* of A .

Let V denote the vector space of complex-valued functions on A . For each $a \in A$ we can define a linear transformation T_a on V by

$$(1.6) \quad T_a(f)(x) = f(x - a)$$

for each $f \in V$. In other words, T_a translates a given function by a . Notice that

$$(1.7) \quad T_a \circ T_b = T_{a+b}$$

for all $a, b \in A$, and that for each $a \in A$ T_a is an invertible linear operator on V , with

$$(1.8) \quad (T_a)^{-1} = T_{-a}.$$

If $\phi \in A^*$, then

$$(1.9) \quad T_a(\phi) = \overline{\phi(a)} \phi,$$

since $\phi(x - a) = \phi(-a)\phi(x)$ and $\phi(-a) = \phi(a)^{-1} = \overline{\phi(a)}$. Thus ϕ is an eigenvector for T_a with eigenvalue $\overline{\phi(a)}$. Now suppose that $f(x)$ is a function on A which is an eigenvector for T_a for each $a \in A$ with eigenvalue $\lambda(a)$, which is to say that

$$(1.10) \quad T_a(f) = \lambda(a) f$$

for all $a \in A$. This is equivalent to

$$(1.11) \quad f(x - a) = \lambda(a) f(x)$$

for all $x, a \in A$. It is easy to see that f is either identically 0 on A , or that λ defines a character on A and f is a nonzero multiple of the complex conjugate of λ .

Let n denote the number of elements of A . If f_1, f_2 are elements of A , then let us define their inner product $\langle f_1, f_2 \rangle_A$ by

$$(1.12) \quad \langle f_1, f_2 \rangle_A = \frac{1}{n} \sum_{x \in A} f_1(x) \overline{f_2(x)}.$$

If f is a function on V , then we also define its norm $\|f\|_A$ by

$$(1.13) \quad \|f\|_A = \left(\frac{1}{n} \sum_{x \in A} |f(x)|^2 \right)^{1/2},$$

which is the same as

$$(1.14) \quad \|f\|_A = \langle f, f \rangle_A^{1/2}.$$

By standard results we have the *Cauchy-Schwarz inequality*

$$(1.15) \quad |\langle f_1, f_2 \rangle_A| \leq \|f_1\|_A \|f_2\|_A$$

and the *triangle inequality*

$$(1.16) \quad \|f_1 + f_2\|_A \leq \|f_1\|_A + \|f_2\|_A$$

for $f_1, f_2 \in A$.

With respect to this inner product, each translation operator T_a is a *unitary* operator on V , so that

$$(1.17) \quad \langle T_a(f_1), T_a(f_2) \rangle_A = \langle f_1, f_2 \rangle_A$$

for all $f_1, f_2 \in A$. It is a well-known result from linear algebra that a unitary linear operator on a finite-dimensional complex inner product space can be diagonalized in an orthonormal basis. Moreover, given a finite collection of unitary transformations which commute with each other, there is in fact an orthonormal basis for the inner product space in which all of the unitary transformations are diagonalized. Of course the eigenvalues of a unitary transformation automatically have modulus equal to 1.

Suppose that ϕ is a character on A . If a is any element of A , then

$$(1.18) \quad \phi(a) \sum_{x \in A} \phi(x) = \sum_{x \in A} \phi(x + a) = \sum_{x \in A} \phi(x),$$

where the second equality holds because the two sums have the same terms, just arranged differently. It follows that either $\phi(a) = 1$ for all $a \in A$, so that ϕ is the unit character, or that

$$(1.19) \quad \sum_{x \in A} \phi(x) = 0.$$

If $\psi_1, \psi_2 \in A^*$, then we can apply this to the character

$$(1.20) \quad \phi(x) = \psi_1(x) \overline{\psi_2(x)}$$

to conclude that either $\psi_1 = \psi_2$ or

$$(1.21) \quad \langle \psi_1, \psi_2 \rangle_A = 0,$$

which is to say that ψ_1, ψ_2 are orthogonal to each other. Of course this also follows from the characterization of characters as simultaneous eigenvectors of the translation operators, since distinct characters correspond to distinct eigenvalues for at least one eigenvector. Because of the way that we defined the inner product on V , characters automatically have norm equal to 1.

Theorem 1.22 *The dual group A^* has the same number of elements as A does, and the elements of A^* form an orthonormal basis for V .*

More precisely, we have seen that the characters are orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle_A$, and hence that they are linearly independent, which implies that there are at most n elements of A^* . We also know that simultaneous eigenvectors of the translation operators are multiples of characters, and since there is an orthonormal basis of simultaneous eigenvectors, we get an orthonormal basis of characters by multiplying the simultaneous eigenvectors in the orthonormal basis by complex numbers with modulus 1, if necessary. In particular, the number of characters is exactly equal to n .

Let W denote the vector space of complex-valued functions on A^* , which therefore has the same dimension as V , n . If h_1, h_2 are elements of W , let us define their inner product to be

$$(1.23) \quad \langle h_1, h_2 \rangle_{A^*} = \sum_{\phi \in A^*} h_1(\phi) \overline{h_2(\phi)}.$$

This leads to the norm

$$(1.24) \quad \|h\|_{A^*} = \langle h, h \rangle_{A^*}^{1/2} = \left(\sum_{\phi \in A^*} |h(\phi)|^2 \right)^{1/2}.$$

As before we have the Cauchy-Schwarz and triangle inequalities, which is to say that

$$(1.25) \quad |\langle h_1, h_2 \rangle_{A^*}| \leq \|h_1\|_{A^*} \|h_2\|_{A^*}$$

and

$$(1.26) \quad \|h_1 + h_2\|_{A^*} \leq \|h_1\|_{A^*} + \|h_2\|_{A^*}$$

for all $h_1, h_2 \in W$.

If f is a function on A , then the *Fourier transform* of f is the function $\widehat{f}(\phi)$ on A^* defined by

$$(1.27) \quad \widehat{f}(\phi) = \langle f, \phi \rangle_A$$

for $\phi \in A$. We may also denote the Fourier transform of f by $\mathcal{F}(f)$, so that \mathcal{F} is a linear transformation from V to W . The orthonormality of the characters on A implies that the Fourier transform is actually a unitary transformation from V to W with respect to the inner products that we have defined, i.e.,

$$(1.28) \quad \langle \mathcal{F}(f_1), \mathcal{F}(f_2) \rangle_{A^*} = \langle f_1, f_2 \rangle_A$$

for all $f_1, f_2 \in V$, and thus

$$(1.29) \quad \|\mathcal{F}(f)\|_{A^*} = \|f\|_A$$

for all $f \in V$.

As a Fourier inversion formula we can write

$$(1.30) \quad f = \sum_{\phi \in A^*} \widehat{f}(\phi) \phi$$

for all $f \in V$. In particular a function f on A is uniquely determined by its Fourier transform, which is to say that the Fourier transform is a one-to-one linear mapping from V into W . It follows that the Fourier transform maps V onto W , since they have the same dimension. In other words, every function on A^* arises as the Fourier transform of a function on A , which can be obtained simply by using the function on A^* as the coefficients for an expansion in characters of a function on A .

If $f_1, f_2 \in V$, then the *convolution* of f_1 and f_2 is the function on A defined by

$$(1.31) \quad (f_1 * f_2)(x) = \frac{1}{n} \sum_{x \in A} f_1(y) f_2(x - y).$$

This is clearly linear in each of f_1 and f_2 . It is also commutative and associative, which is to say that

$$(1.32) \quad f_1 * f_2 = f_2 * f_1$$

and

$$(1.33) \quad (f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$$

for all $f_1, f_2, f_3 \in V$. Moreover,

$$(1.34) \quad \mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1) \mathcal{F}(f_2)$$

for all $f_1, f_2 \in V$, i.e., the Fourier transform of a convolution is the same as the product of the individual Fourier transforms.

2 Riesz–Thorin convexity

Fix a positive integer m , and consider \mathbf{C}^m as an m -dimensional vector space over the complex numbers. If p is a real number such that $1 \leq p < \infty$ and $v = (v_1, \dots, v_m)$ is a vector in \mathbf{C}^m , put

$$(2.1) \quad \|v\|_p = \left(\sum_{j=1}^m |v_j|^p \right)^{1/p}.$$

When $p = \infty$ set

$$(2.2) \quad \|v\|_\infty = \max\{|v_j| : 1 \leq j \leq m\}.$$

For $1 \leq p \leq \infty$ and $v \in \mathbf{C}^m$ we have that $\|v\|_p$ is a nonnegative real number which is equal to 0 if and only if $v = 0$, and that

$$(2.3) \quad \|\alpha v\|_p = |\alpha| \|v\|_p$$

for all complex numbers α . The triangle inequality

$$(2.4) \quad \|v + w\|_p \leq \|v\|_p + \|w\|_p$$

for all $v, w \in \mathbf{C}^m$ is equivalent to the convexity of the closed unit ball

$$(2.5) \quad B_p = \{v \in \mathbf{C}^m : \|v\|_p \leq 1\},$$

which is to say that $t v + (1 - t) w \in B_p$ whenever $v, w \in B_p$ and $0 \leq t \leq 1$. When $p = 1, \infty$ it is quite easy to verify the triangle inequality directly, while for $1 < p < \infty$ one can derive it from the convexity of the function x^p on $[0, \infty)$.

It is clear from the definitions that

$$(2.6) \quad \|v\|_\infty \leq \|v\|_p$$

when $v \in \mathbf{C}^m$ and $1 \leq p < \infty$. Using this one can check that

$$(2.7) \quad \|v\|_q \leq \|v\|_p$$

when $v \in \mathbf{C}^m$ and $1 \leq p \leq q < \infty$. Namely,

$$(2.8) \quad \begin{aligned} \|v\|_q^q = \sum_{j=1}^m |v_j|^q &\leq \|v\|_\infty^{q-p} \sum_{j=1}^m |v_j|^p \\ &\leq \|v\|_p^{q-p} \|v\|_p^p = \|v\|_p^q. \end{aligned}$$

In the other direction,

$$(2.9) \quad \|v\|_p \leq m^{1/p} \|v\|_\infty$$

when $v \in \mathbf{C}^m$ and $1 \leq p < \infty$. This follows easily from the definitions. A slightly more tricky fact is that

$$(2.10) \quad \|v\|_p \leq m^{1/p-1/q} \|v\|_\infty$$

for $1 \leq p \leq q < \infty$. This can be derived from the convexity of the function $x^{q/p}$ on $[0, \infty)$.

If $1 \leq p \leq \infty$, then the *conjugate exponent* p' , $1 \leq p' \leq \infty$, is defined by the condition

$$(2.11) \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Hölder's inequality states that

$$(2.12) \quad \left| \sum_{j=1}^m v_j w_j \right| \leq \|v\|_p \|w\|_{p'}$$

for all $v, w \in \mathbf{C}^m$. When $p = 1, p' = \infty$ or $p = \infty, p' = 1$ this is a straightforward consequence of the definitions, and when $p = p' = 2$ this is the classical Cauchy–Schwarz inequality.

Suppose that $1 < p < \infty$, so that $1 < p' < \infty$, and let us prove Hölder's inequality. If x, y are nonnegative real numbers, then we have that

$$(2.13) \quad x y \leq \frac{x^p}{p} + \frac{y^{p'}}{p'}.$$

This can be derived from the convexity of the exponential function, for instance. For $v, w \in \mathbf{C}^m$ it follows that

$$(2.14) \quad \left| \sum_{j=1}^m v_j w_j \right| \leq \sum_{j=1}^m |v_j| |w_j| \leq \frac{\|v\|_p^p}{p} + \frac{\|w\|_{p'}^{p'}}{p'},$$

by applying the previous inequality to each of the terms in the sum. This implies Hölder's inequality when $\|v\|_p, \|w\|_{p'}$ are equal to 1, and one can reduce to that case using the homogeneity of the norms.

Hölder's inequality is sharp, in the sense that for each $v \in \mathbf{C}^m$ and $1 \leq p \leq \infty$ there is a $w \in \mathbf{C}^m$ such that $\|w\|_{p'} = 1$ and

$$(2.15) \quad \sum_{j=1}^m v_j w_j = \|v\|_p.$$

This is easy to verify from the definitions. Notice that if $1 < p < \infty$ and $v \neq 0$, then w is unique.

Now let T be a linear transformation from \mathbf{C}^m into itself. Suppose that

$$(2.16) \quad 1 \leq p_0, p_1, q_0, q_1 \leq \infty,$$

and let p'_0, p'_1, q'_0, q'_1 be their conjugate exponents. Suppose also that L_0, L_1 are nonnegative real numbers such that

$$(2.17) \quad \|T(v)\|_{q_0} \leq L_0 \|v\|_{p_0}$$

and

$$(2.18) \quad \|T(v)\|_{q_1} \leq L_1 \|v\|_{p_1}$$

for all $v \in \mathbf{C}^m$. The *Riesz–Thorin convexity theorem* states that

$$(2.19) \quad \|T(v)\|_{q_t} \leq L_t \|v\|_{p_t}$$

for $0 < t < 1$, where $1/p_t = (1-t)/p_0 + t/p_1$, $1/q_t = (1-t)/q_0 + t/q_1$, and $L_t = L_0^{(1-t)} L_1^t$.

Here is an equivalent formulation, using Hölder's inequality and the fact that equality is attained in Hölder's inequality. Suppose that

$$(2.20) \quad \left| \sum_{j=1}^m (T(v))_j w_j \right| \leq L_0 \|v\|_{p_0} \|w\|_{q'_0}$$

and

$$(2.21) \quad \left| \sum_{j=1}^m (T(v))_j w_j \right| \leq L_1 \|v\|_{p_1} \|w\|_{q'_1}$$

for all $v, w \in \mathbf{C}^m$, where $(T(v))_j$ denotes the j th component of $T(v)$. Then

$$(2.22) \quad \left| \sum_{j=1}^m (T(v))_j w_j \right| \leq L_t \|v\|_{p_t} \|w\|_{q'_t}$$

for all $v, w \in \mathbf{C}^m$ and $0 < t < 1$, with q'_t being the conjugate exponent of q_t .

Marcel Riesz proved this originally using a real-variable method that worked for both \mathbf{R}^m and \mathbf{C}^m , and with the additional hypothesis that $q_0 \geq p_0, q_1 \geq p_1$.

Thorin then found a way to use complex analytic functions to prove this in a nice way and without this extra condition on the exponents, in the complex case. In general the real case can be reduced to the complex case with a slightly less sharp inequality. We shall now discuss Thorin's argument.

We begin with some preliminary facts from complex analysis. Let U be a nonempty bounded open subset of the complex plane, and let $f(z)$ be a continuous complex-valued function on the closure \overline{U} of U which is complex-analytic on U . Because U is bounded, \overline{U} and ∂U are closed and bounded subsets of the complex plane which are therefore compact, and the maximum of $|f(z)|$ on \overline{U} and ∂U are attained. The *maximum principle* for complex analytic functions implies that

$$(2.23) \quad \max\{|f(z)| : z \in \overline{U}\} \leq \max\{|f(z)| : z \in \partial U\}.$$

We would like to apply this to the case where U is the strip $\{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\}$, where $\operatorname{Re} z$ denotes the real part of a complex number z . This region is not bounded, and we shall deal with that in a moment. Let $f(z)$ be a continuous complex-valued function defined on the closure of U , which is the set of $z \in \mathbf{C}$ such that $0 \leq \operatorname{Re} z \leq 1$, and suppose that $f(z)$ is holomorphic on U and bounded on the closure of U . For $0 \leq t \leq 1$ put

$$(2.24) \quad M_t = \sup\{|f(z)| : \operatorname{Re} z = t\}.$$

We would like to say that

$$(2.25) \quad M_t \leq \max(M_0, M_1)$$

when $0 < t < 1$, which would say that the supremum of $|f(z)|$ over the closure of U is equal to the supremum of $|f(z)|$ over the boundary of U .

If $|f(z)| \rightarrow 0$ as $|z| \rightarrow \infty$, then this extension of the maximum principle can be derived from the version for bounded regions, by approximating the strip U with large rectangular subsets. For a general bounded function $f(z)$ we can reduce to this case using the following trick. For each $\epsilon > 0$, consider the function

$$(2.26) \quad f_\epsilon(z) = \exp(\epsilon z^2) f(z).$$

This is also a continuous complex-valued function on the closure of U which is complex-analytic on U . If $z = x + iy$, with $x, y \in \mathbf{R}$, then

$$(2.27) \quad |\exp(\epsilon z^2)| = \exp(\epsilon(x^2 - y^2)),$$

and one can derive the maximum principle for f on the closure of U from the corresponding statement for f_ϵ for all $\epsilon > 0$.

As a refinement of the maximum principle in this situation there is the *three lines theorem*, which states that if $f(z)$ is a bounded continuous complex-analytic function on the closure of U which is complex-analytic on U , and if M_t is as defined previously, then

$$(2.28) \quad M_t \leq M_0^{(1-t)} M_1^t$$

when $0 < t < 1$. This can be derived by applying the previous version to functions of the form $\exp(az) f(z)$, where a is a real number. The main point is that

$$(2.29) \quad |\exp(az)| = \exp(ax)$$

if $x = \operatorname{Re} z$, and in particular this quantity is constant on the lines $\operatorname{Re} z = t$.

Now let us return to the Riesz–Thorin convexity theorem. Let $v, w \in \mathbf{C}^m$ and $0 < t < 1$ be given. Without loss of generality let us make the normalizing assumption that

$$(2.30) \quad \|v\|_{p_t} = \|w\|_{q'_t} = 1.$$

We would like to show that

$$(2.31) \quad \left| \sum_{j=1}^m (T(v))_j w_j \right| \leq L_0^{(1-t)} L_1^t.$$

The idea is to realize this as a case of the three lines theorem. To do this we would like to find bounded continuous \mathbf{C}^m -valued functions $\alpha(z)$, $\beta(z)$ defined on the closed unit strip $\{z \in \mathbf{C} : 0 \leq \operatorname{Re} z \leq 1\}$ which are complex-analytic in the interior and satisfy

$$(2.32) \quad \alpha(t) = v, \quad \beta(t) = w$$

and

$$(2.33) \quad \|\alpha(z)\|_{p_x} = \|\beta(z)\|_{q'_x} = 1$$

when $0 \leq x \leq 1$ and $\operatorname{Re} z = x$. If we can do this, then the desired inequality follows from the three lines theorem applied to

$$(2.34) \quad \sum_{j=1}^m (T(\alpha(z)))_j \beta_j(z).$$

Here and in the following we write $\alpha_j(z)$, $\beta_j(z)$ for the components of $\alpha(z)$, $\beta(z)$.

When $v_j = 0$ or $w_k = 0$ we put $\alpha_j(z) = 0$ and $\beta_k(z) = 0$ for all z . Otherwise we put

$$(2.35) \quad \alpha_j(z) = v_j |v_j|^{a_0 z + a_1}, \quad \beta_k(z) = w_k |w_k|^{b_0 z + b_1},$$

where a_0 , a_1 , b_0 , b_1 are fixed real numbers. More precisely, they should be chosen so that

$$(2.36) \quad \frac{1}{p_t}(a_0 x + a_1 + 1) = \frac{1}{p_x}, \quad \frac{1}{q'_t}(b_0 x + b_1 + 1) = \frac{1}{q'_x}$$

when $0 \leq x \leq 1$. The details are left as an exercise to the interested reader.

Now let us consider applications of the Riesz–Thorin theorem to Fourier analysis on finite abelian groups as in the previous section. Let A be a finite abelian group with n elements, and let A^* be the dual group. Also let V denote the vector space of complex-valued functions on A , and let W denote the vector space of complex-valued functions on A^* , each of which can be identified with \mathbf{C}^n .

If $f \in V$, put

$$(2.37) \quad \|f\|_{p,A} = \left(\frac{1}{n} \sum_{x \in A} |f(x)|^p \right)^{1/p}$$

when $1 \leq p < \infty$ and

$$(2.38) \quad \|f\|_{\infty,A} = \max\{|f(x)| : x \in A\}.$$

Thus $\|f\|_{p,A}$ differs from the norm previously defined by a constant factor, which does not really cause any trouble. For $h \in W$ we put

$$(2.39) \quad \|h\|_{p,A^*} = \left(\sum_{\phi \in A^*} |h(\phi)|^p \right)^{1/p}$$

when $1 \leq p < \infty$ and

$$(2.40) \quad \|h\|_{\infty,A^*} = \max\{|h(\phi)| : \phi \in A^*\},$$

which is exactly the same as the norms defined earlier in this section.

For the record, if $f \in V$ then

$$(2.41) \quad \|f\|_{p,A} \leq \|f\|_{q,A} \leq n^{1/p-1/q} \|f\|_{p,A}$$

when $1 \leq p \leq q \leq \infty$. For $h \in W$ we still have

$$(2.42) \quad \|h\|_{q,A^*} \leq \|h\|_{p,A^*} \leq n^{1/p-1/q} \|h\|_{q,A^*}$$

when $1 \leq p \leq q \leq \infty$. We can write Hölder's inequality as

$$(2.43) \quad \frac{1}{n} \left| \sum_{x \in A} f_1(x) f_2(x) \right| \leq \|f_1\|_{p,A} \|f_2\|_{p',A}$$

for all $f_1, f_2 \in A$ and

$$(2.44) \quad \left| \sum_{\phi \in A^*} h_1(\phi) h_2(\phi) \right| \leq \|h_1\|_{p,A^*} \|h_2\|_{p',A^*}$$

for all $h_1, h_2 \in W$. Here $1 \leq p \leq \infty$ and p' is the conjugate exponent of p .

Let $\mathcal{F} : V \rightarrow W$ denote the Fourier transform. Thus

$$(2.45) \quad \|\mathcal{F}(f)\|_{2,A^*} = \|f\|_{2,A}$$

for all $f \in V$, since when $p = 2$ the norms defined here reduce to the ones discussed in the previous section, associated to the inner products on V, W . It is easy to see that

$$(2.46) \quad \|\mathcal{F}(f)\|_{\infty,A^*} \leq \|f\|_{1,A}$$

for all $f \in V$, since the characters on A have modulus equal to 1 at every point in A . The convexity theorem implies that

$$(2.47) \quad \|\mathcal{F}(f)\|_{p',A^*} \leq \|f\|_{p,A}$$

for all $f \in V$ when $1 \leq p \leq 2$.

Now let us consider some inequalities related to convolutions. Let f_1, f_2 be functions on V . If $1 \leq p \leq \infty$, then one can check that

$$(2.48) \quad \|f_1 * f_2\|_{p,A} \leq \|f_1\|_{1,A} \|f_2\|_{p,A}.$$

Indeed, one can think of the convolution $f_1 * f_2$ as being a linear combination of translates of f_2 with coefficients given by the values of f_1 . The translates of f_2 have the same $\|\cdot\|_{p,A}$ norm as f_2 , so that the $\|\cdot\|_{p,A}$ norm of $f_1 * f_2$ can be estimated as above using the homogeneity and triangle inequality for the norm.

Using Hölder's inequality it follows that

$$(2.49) \quad \|f_1 * f_2\|_{\infty,A} \leq \|f_1\|_{p',A} \|f_2\|_{p,A}.$$

Let us think of f_2 as being fixed, and

$$(2.50) \quad f_1 \mapsto f_1 * f_2$$

as being a linear mapping from V to itself. Let us also fix p , so that we have two estimates for this linear mapping as in the preceding inequalities. From the convexity theorem it follows that if $1 \leq r \leq p'$ and q is defined by

$$(2.51) \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1,$$

then

$$(2.52) \quad \|f_1 * f_2\|_{q,A} \leq \|f_1\|_{r,A} \|f_2\|_{p,A}.$$

Fix a function b on A , and define a linear transformation $T_b : V \rightarrow V$ by

$$(2.53) \quad T_b(f) = f * b.$$

As above, if $1 \leq p \leq \infty$, then

$$(2.54) \quad \|T_b(f)\|_{p,A} \leq \|b\|_{1,A} \|f\|_{p,A}.$$

When $p = 1$ this is sharp, because if δ_0 denotes the function on A such that $\delta_0(x) = 1$ when $x = 0$ and $\delta_0(x) = 0$ when $x \neq 0$, then

$$(2.55) \quad T_b(\delta_0) = b$$

and

$$(2.56) \quad \|T_b\|_{1,A} = \|b\|_{1,A} = \|b\|_{1,A} \|\delta_0\|_{1,A}.$$

Now suppose that $p = 2$. For this we use the Fourier transform, the fact that it converts convolutions into multiplications, and the unitary property for the norms. Namely,

$$(2.57) \quad \mathcal{F}(T_b(f)) = \mathcal{F}(b) \mathcal{F}(f)$$

and

$$(2.58) \quad \|\mathcal{F}(T_b(f))\|_{2,A^*} = \|T_b(f)\|_{2,A},$$

and therefore

$$(2.59) \quad \|T_b(f)\|_{2,A} \leq \|\mathcal{F}(b)\|_{\infty,A^*} \|f\|_{2,A}.$$

To put it another way, T_b is diagonalized by the orthonormal basis of V consisting of the characters, with the diagonal entries given by the values of the Fourier transform of b . This inequality is also sharp, with equality being attained for characters ϕ on A such that

$$(2.60) \quad |\mathcal{F}(\phi)|$$

is as large as possible.

We can apply the convexity theorem to these two estimates for $p = 1$ and $p = 2$ to obtain that

$$(2.61) \quad \|T_b(f)\|_{p,A} \leq \|b\|_{1,A}^{2/p-1} \|\mathcal{F}(b)\|_{\infty,A^*}^{2-2/p} \|f\|_{p,A}$$

when $1 \leq p \leq 2$. When $p \geq 2$ the estimate is the same as for p' , $1 \leq p' \leq 2$, by a duality argument. To be more precise, if p is fixed and M is a nonnegative real number such that

$$(2.62) \quad \|T_b(f)\|_{p,A} \leq M \|f\|_{p,A}$$

for all $f \in V$, then the analogous inequality holds also for p' , i.e.,

$$(2.63) \quad \|T_b(f)\|_{p',A} \leq M \|f\|_{p',A}$$

for all $f \in V$.

Indeed, if f_1, f_2 are functions on A , then

$$(2.64) \quad \begin{aligned} \frac{1}{n} \sum_{x \in A} T_b(f_1)(x) f_2(x) &= \frac{1}{n^2} \sum_{x \in A} \sum_{y \in A} b(y) f_1(x-y) f_2(x) \\ &= \frac{1}{n^2} \sum_{x \in A} \sum_{y \in A} b(x-y) f_1(y) f_2(x) \\ &= \frac{1}{n} \sum_{y \in A} f_1(y) \tilde{T}_b(f_2)(y), \end{aligned}$$

where

$$(2.65) \quad \tilde{b}(z) = b(-z).$$

Using this one can check that

$$(2.66) \quad \|T_b(f_1)\|_{p,A} \leq M \|f_1\|_{p,A}$$

for all $f_1 \in V$ holds if and only if

$$(2.67) \quad \|\tilde{T}_b(f_2)\|_{p',A} \leq M \|f_2\|_{p',A}$$

for all $f_2 \in V$. Also, for any q ,

$$(2.68) \quad \|\tilde{T}_b(f)\|_{q,A} \leq M \|f\|_{q,A}$$

holds for all $f \in V$ if and only if

$$(2.69) \quad \|T_b(f)\|_{q,A} \leq M \|f\|_{q,A}.$$

This follows from the fact that

$$(2.70) \quad T_b^*(f)(-x) = T_b(\tilde{f})(x).$$

3 Continuous functions on \mathbf{R}^n

Recall that if $x = (x_1, \dots, x_n)$ is an element of \mathbf{R}^n , then the standard Euclidean norm of x is defined by

$$(3.1) \quad |x| = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.$$

If x, y are elements of \mathbf{R}^n , then their inner product is defined by

$$(3.2) \quad \langle x, y \rangle = \sum_{j=1}^n x_j y_j,$$

and we have that

$$(3.3) \quad |x| = \langle x, x \rangle^{1/2}.$$

The Cauchy–Schwarz inequality states that

$$(3.4) \quad |\langle x, y \rangle| \leq |x| |y|$$

for all $x, y \in \mathbf{R}^n$, and the triangle inequality states that

$$(3.5) \quad |x + y| \leq |x| + |y|.$$

The standard Euclidean distance on \mathbf{R}^n is defined by

$$(3.6) \quad d(x, y) = |x - y|.$$

Let $C(\mathbf{R}^n)$ denote the vector space of complex-valued continuous functions on \mathbf{R}^n . As usual, sums and products of continuous functions are continuous, and polynomials are continuous. If $\{f_l\}_{l=1}^\infty$ is a sequence of continuous functions on \mathbf{R}^n and f is another function on \mathbf{R}^n , then we say that $\{f_l\}_{l=1}^\infty$ converges to f uniformly on compact sets if for each compact subset K of \mathbf{R}^n and each $\epsilon > 0$ there is a positive integer L such that

$$(3.7) \quad |f_l(x) - f(x)| < \epsilon$$

for all $x \in K$ and $l \geq L$. It follows from standard results in analysis that f is also a continuous function in this case. If two sequences of continuous functions on \mathbf{R}^n converge uniformly on compact subsets of \mathbf{R}^n , then the sums and products of the functions in the two sequences also converge uniformly on

compact subsets, to the sum and product of the limits of the original sequences, respectively.

Let f be a continuous function on \mathbf{R}^n . The *support* of f is denoted $\text{supp } f$ and defined to be the closure of the set of $x \in \mathbf{R}^n$ such that $f(x) \neq 0$. This is equivalent to saying that the complement of the support of f in \mathbf{R}^n consists of the points $x \in \mathbf{R}^n$ such that f vanishes on a neighborhood of x . We write $C_{00}(\mathbf{R}^n)$ for the vector space of continuous functions on \mathbf{R}^n with compact support, which is a vector subspace of $C(\mathbf{R}^n)$.

A continuous function f on \mathbf{R}^n is said to vanish at infinity if for every $\epsilon > 0$ there is a compact subset K of \mathbf{R}^n such that

$$(3.8) \quad |f(x)| < \epsilon$$

for all $x \in \mathbf{R}^n \setminus K$. The vector space of continuous functions on \mathbf{R}^n which vanish at infinity is denoted $C_0(\mathbf{R}^n)$. A continuous function f on \mathbf{R}^n is said to be bounded if there is a nonnegative real number M such that

$$(3.9) \quad |f(x)| \leq M$$

for all $x \in \mathbf{R}^n$. The vector space of bounded continuous functions on \mathbf{R}^n is denoted $C_b(\mathbf{R}^n)$. Observe that the product of a continuous function on \mathbf{R}^n which vanishes at infinity and a bounded continuous function on \mathbf{R}^n also vanishes at infinity.

Let us say that a continuous function f on \mathbf{R}^n has at most polynomial growth if there is a nonnegative real number C and a nonnegative integer k such that

$$(3.10) \quad |f(x)| \leq C(1 + |x|^k)$$

for all $x \in \mathbf{R}^n$. The vector space of continuous functions with at most polynomial growth is denoted $C_p(\mathbf{R}^n)$. Thus we have the inclusions

$$(3.11) \quad C_{00}(\mathbf{R}^n) \subseteq C_0(\mathbf{R}^n) \subseteq C_b(\mathbf{R}^n) \subseteq C_p(\mathbf{R}^n) \subseteq C(\mathbf{R}^n).$$

For $f \in C_b(\mathbf{R}^n)$ we can define the supremum norm

$$(3.12) \quad \|f\| = \sup\{|f(x)| : x \in \mathbf{R}^n\},$$

and this satisfies

$$(3.13) \quad \|f_1 + f_2\| \leq \|f_1\| + \|f_2\|, \quad \|f_1 f_2\| \leq \|f_1\| \|f_2\|$$

for all $f_1, f_2 \in C_b(\mathbf{R}^n)$. The supremum metric on $C_b(\mathbf{R}^n)$ is defined by

$$(3.14) \quad \sigma(f_1, f_2) = \|f_1 - f_2\|.$$

Convergence of a sequence of functions in $C_b(\mathbf{R}^n)$ with respect to the supremum metric is equivalent to classical uniform convergence on all of \mathbf{R}^n . It is well known that $C_b(\mathbf{R}^n)$ is complete as a metric space when equipped with the

supremum norm, which is to say that every Cauchy sequence in $C_b(\mathbf{R}^n)$ with respect to the supremum norm converges in $C_b(\mathbf{R}^n)$.

It is also well known that if a sequence of continuous functions on \mathbf{R}^n which vanish at infinity converges uniformly on \mathbf{R}^n , then the limit also vanishes at infinity. In other words, $C_0(\mathbf{R}^n)$ is a closed linear subspace of $C_b(\mathbf{R}^n)$ with respect to the supremum metric. Moreover, $C_{00}(\mathbf{R}^n)$ is dense in $C_0(\mathbf{R}^n)$ with respect to the supremum metric, which is equivalent to saying that if f is a continuous function on \mathbf{R}^n which vanishes at infinity, then there is a sequence of continuous functions on \mathbf{R}^n with compact support which converge to f uniformly. These approximations of f can be obtained by multiplying f by continuous functions which are equal to 1 on large compact sets, have compact support, and have supremum norm equal to 1.

Let us say that a sequence of functions $\{f_l\}_{l=1}^\infty$ in $C_b(\mathbf{R}^n)$ converges to another function f in $C_b(\mathbf{R}^n)$ in the restricted sense if the supremum norms of the f_l 's are all bounded, and if the f_l 's converge to f uniformly on compact subsets of \mathbf{R}^n . Of course this holds if the f_l 's converge to f uniformly on all of \mathbf{R}^n . Restricted convergence is more general, however, and indeed if f is any bounded continuous function on \mathbf{R}^n , then there is a sequence of continuous functions on \mathbf{R}^n with compact support which converges to f in the restricted sense. Similarly, if f is any continuous function on \mathbf{R}^n , then there is a sequence of continuous functions on \mathbf{R}^n with compact supports which converges to f uniformly on compact subsets. These approximations of f can be obtained by multiplying f by the same kind of functions with compact support as in the previous paragraph.

There is a natural kind of restricted convergence in $C_p(\mathbf{R}^n)$, which is to say that a sequence $\{f_l\}_{l=1}^\infty$ of functions in $C_p(\mathbf{R}^n)$ converges to a function f in this sense if there exist $C, k \geq 0$ such that

$$(3.15) \quad |f_l(x)| \leq C(1 + |x|^k)$$

for all $x \in \mathbf{R}^n$ and $l \geq 1$, and if $\{f_l\}_{l=1}^\infty$ converges to f uniformly on compact sets. This is equivalent to saying that there is a $k \geq 0$ such that

$$(3.16) \quad \frac{f_l - f}{1 + |x|^{k+1}}$$

converges uniformly to 0 on \mathbf{R}^n as $l \rightarrow \infty$. Of course, if a sequence of functions $\{f_l\}_{l=1}^\infty$ in $C_p(\mathbf{R}^n)$ converges to a function f on \mathbf{R}^n in this restricted sense, then f is also in $C_p(\mathbf{R}^n)$. Every function in $C_p(\mathbf{R}^n)$ is the limit of a sequence of compactly supported continuous functions in this restricted sense.

Let us mention also a notion of restricted convergence in $C_{00}(\mathbf{R}^n)$, which is that a sequence $\{f_l\}_{l=1}^\infty$ of functions in $C_{00}(\mathbf{R}^n)$ converges to a function f on \mathbf{R}^n in this sense if there is a single compact subset K of \mathbf{R}^n such that the support of f_l is contained in K for all l , and if $\{f_l\}_{l=1}^\infty$ converges to f uniformly. In this case the support of f is also contained in K , so that $f \in C_{00}(\mathbf{R}^n)$. Under these conditions uniform convergence is equivalent to uniform convergence on K , with f set to 0 on $\mathbf{R}^n \setminus K$.

Let us write $\mathcal{M}(\mathbf{R}^n)$ for the vector space of complex-linear functionals on $C_{00}(\mathbf{R}^n)$ which are bounded on compact subsets of \mathbf{R}^n . More precisely, an element λ of $\mathcal{M}(\mathbf{R}^n)$ is a linear mapping from $C_{00}(\mathbf{R}^n)$ into the complex numbers with the property that for each compact subset K of \mathbf{R}^n , there is a nonnegative real number $L(K)$ such that

$$(3.17) \quad |\lambda(f)| \leq L(K) \|f\|$$

whenever $f \in C_{00}(\mathbf{R}^n)$ has support contained in K . Informally we might refer to the elements of $\mathcal{M}(\mathbf{R}^n)$ as measures on \mathbf{R}^n . We shall write $\mathcal{M}_r(\mathbf{R}^n)$ for the real measures on \mathbf{R}^n , which is to say the $\lambda \in \mathcal{M}(\mathbf{R}^n)$ such that $\lambda(f)$ is a real number whenever f is a real-valued continuous function on \mathbf{R}^n .

If $\lambda \in \mathcal{M}(\mathbf{R}^n)$, then λ is a continuous linear functional on $C_{00}(\mathbf{R}^n)$ with respect to the kind of restricted convergence defined before. Namely, if K is a compact subset of \mathbf{R}^n , $\{f_l\}_{l=1}^\infty$ is a sequence of continuous functions on \mathbf{R}^n with supports contained in K , and if f is a continuous function on \mathbf{R}^n with support contained in K such that $\{f_l\}_{l=1}^\infty$ converges uniformly to f , then

$$(3.18) \quad \lim_{l \rightarrow \infty} \lambda(f_l) = \lambda(f).$$

This is easy to derive from the boundedness of λ on compact subsets of \mathbf{R}^n .

Let $\mathcal{M}(\mathbf{R}^n)$ denote the set of nonnegative linear functionals on $C_{00}(\mathbf{R}^n)$, which is to say the linear mappings λ from $C_{00}(\mathbf{R}^n)$ into the complex numbers such that for each nonnegative real-valued function f in $C_{00}(\mathbf{R}^n)$ we have that $\lambda(f)$ is a real number and

$$(3.19) \quad \lambda(f) \geq 0.$$

If $\lambda \in \mathcal{M}_+(\mathbf{R}^n)$, then $\lambda(f)$ is a real number for every real-valued function f in $C_{00}(\mathbf{R}^n)$, because every such function can be written as $f_1 - f_2$ where f_1, f_2 are nonnegative real-valued functions in $C_{00}(\mathbf{R}^n)$.

Suppose that $\lambda \in \mathcal{M}_+(\mathbf{R}^n)$ and that ϕ, f are functions in $C_{00}(\mathbf{R}^n)$ such that ϕ is real-valued and

$$(3.20) \quad |f(x)| \leq \phi(x)$$

for all $x \in \mathbf{R}^n$. This is equivalent to saying that

$$(3.21) \quad \operatorname{Re}(\phi(x) - \theta f(x)) \geq 0$$

for all complex numbers θ with $|\theta| = 1$ and all $x \in \mathbf{R}^n$. In this case we get that

$$(3.22) \quad \operatorname{Re}(\lambda(\phi) - \theta \lambda(f)) \geq 0$$

for all complex numbers θ such that $|\theta| = 1$, and therefore that

$$(3.23) \quad |\lambda(f)| \leq \lambda(\phi)$$

in this case.

In particular it follows that λ is bounded on compact subsets of \mathbf{R}^n , and hence that $\lambda \in \mathcal{M}_r(\mathbf{R}^n)$. Indeed, if K is a compact subset of \mathbf{R}^n and λ is a

nonnegative real-valued function in $C_{00}(\mathbf{R}^n)$ such that $\phi(x) = 1$ for all $x \in K$, then one can check that

$$(3.24) \quad |\lambda(f)| \leq \lambda(\phi) \|f\|$$

for all $f \in C_{00}(\mathbf{R}^n)$ with support contained in K . Notice that $\mathcal{M}_r(\mathbf{R}^n)$ is a real vector space, and that $\mathcal{M}_+(\mathbf{R}^n)$ is a cone in $\mathcal{M}(\mathbf{R}^n)$, in the sense that if $\lambda_1, \lambda_2 \in \mathcal{M}_+(\mathbf{R}^n)$ and a_1, a_2 are nonnegative real numbers, then $a_1 \lambda_1 + a_2 \lambda_2 \in \mathcal{M}_+(\mathbf{R}^n)$. The elements of $\mathcal{M}(\mathbf{R}^n)$ can be described informally as nonnegative measures on \mathbf{R}^n .

Suppose that U is an open subset of \mathbf{R}^n , and that $\lambda \in \mathcal{M}(\mathbf{R}^n)$. We say that λ vanishes on U if $\lambda(f) = 0$ whenever $f \in C_{00}(\mathbf{R}^n)$ has support contained in U . If U_1, U_2 are open subsets of \mathbf{R}^n , $\lambda \in \mathcal{M}(\mathbf{R}^n)$, and λ vanishes on U_1, U_2 , then λ vanishes on $U_1 \cup U_2$. For if $f \in C_{00}(\mathbf{R}^n)$ has support contained in $U_1 \cup U_2$, then f can be written as $f_1 + f_2$ where $f_1, f_2 \in C_{00}(\mathbf{R}^n)$ and the supports of f_1, f_2 are contained in U_1, U_2 , respectively, and therefore

$$(3.25) \quad \lambda(f) = \lambda(f_1) + \lambda(f_2) = 0.$$

More generally, if $\{U_\alpha\}_{\alpha \in A}$ is a family of open subsets of \mathbf{R}^n , $\lambda \in \mathcal{M}(\mathbf{R}^n)$, and λ vanishes on each U_α , then λ vanishes on the union of the U_α 's. Indeed, let f be a function in $C_{00}(\mathbf{R}^n)$ whose support is contained in the union of the U_α 's. Because the support of f is compact by assumption, it follows that the support of f is contained in the union of finitely many U_α 's. The result in the previous paragraph may be applied to obtain that $\lambda(f) = 0$.

Let $\lambda \in \mathcal{M}(\mathbf{R}^n)$ be given. The support of λ is defined to be the set of points x in \mathbf{R}^n such that λ does not vanish on a neighborhood of x . In other words, the complement of the support of λ is the set of points $y \in \mathbf{R}^n$ such that λ vanishes on a neighborhood of y . This complementary set is open by construction, and hence the support of λ is automatically a closed subset of \mathbf{R}^n . Also, λ vanishes on the complement of its support in \mathbf{R}^n , by the result mentioned in the previous paragraph.

If $\lambda \in \mathcal{M}(\mathbf{R}^n)$ and $\phi \in C(\mathbf{R}^n)$, then we can define a new measure $\lambda_\phi \in \mathcal{M}(\mathbf{R}^n)$ by

$$(3.26) \quad \lambda_\phi(f) = \lambda(\phi f)$$

for $f \in C_{00}(\mathbf{R}^n)$. The support of λ_ϕ is automatically contained in the support of ϕ as a continuous function on \mathbf{R}^n . A measure $\lambda \in \mathcal{M}(\mathbf{R}^n)$ has compact support if and only

$$(3.27) \quad \lambda = \lambda_\phi$$

for some $\phi \in C_{00}(\mathbf{R}^n)$.

If $\phi \in C_{00}(\mathbf{R}^n)$ and $\lambda \in \mathcal{M}(\mathbf{R}^n)$, then we extend λ_ϕ to a linear functional on all of $C(\mathbf{R}^n)$ in an obvious manner. Specifically, we put

$$(3.28) \quad \lambda_\phi(f) = \lambda(\phi f)$$

for all $f \in C(\mathbf{R}^n)$, and this makes sense because $\phi f \in C_{00}(\mathbf{R}^n)$ since $\phi \in C_{00}(\mathbf{R}^n)$. This extension to $C(\mathbf{R}^n)$ is continuous in the sense that if $\{f_l\}_{l=1}^\infty$

is a sequence of continuous functions on \mathbf{R}^n which converges uniformly to the continuous function f on compact subsets of \mathbf{R}^n , then

$$(3.29) \quad \lim_{l \rightarrow \infty} \lambda_\phi(f_l) = \lambda_\phi(f),$$

essentially because we only need the uniform convergence of the f_l 's to f on the support of ϕ . This continuity property uniquely characterizes the extension of λ_ϕ to $C(\mathbf{R}^n)$, since $C_{00}(\mathbf{R}^n)$ is dense in $C(\mathbf{R}^n)$ with respect to uniform convergence on compact subsets of \mathbf{R}^n .

A measure $\lambda \in \mathcal{M}(\mathbf{R}^n)$ is said to be bounded if there is a nonnegative real number L such that

$$(3.30) \quad |\lambda(f)| \leq L \|f\|$$

for all $f \in C_{00}(\mathbf{R}^n)$. We write $\mathcal{M}_b(\mathbf{R}^n)$ for the vector subspace of $\mathcal{M}(\mathbf{R}^n)$ of bounded measures. For $\lambda \in \mathcal{M}_b(\mathbf{R}^n)$ we define the norm of λ by

$$(3.31) \quad \|\lambda\|_* = \sup\{|\lambda(f)| : f \in C_{00}(\mathbf{R}^n), \|f\| \leq 1\}.$$

This is the same as saying that $\|\lambda\|_*$ is the smallest choice of L for which the previous inequality holds.

It is easy to see that if $\lambda \in \mathcal{M}_b(\mathbf{R}^n)$ and a is a complex number, then

$$(3.32) \quad \|a \lambda\|_* = |a| \|\lambda\|_*.$$

If $\lambda_1, \lambda_2 \in \mathcal{M}_b(\mathbf{R}^n)$, then

$$(3.33) \quad \|\lambda_1 + \lambda_2\|_* \leq \|\lambda_1\|_* + \|\lambda_2\|_*.$$

If $\lambda \in \mathcal{M}_b(\mathbf{R}^n)$ and $\phi \in C_b(\mathbf{R}^n)$, then it is easy to check that $\lambda_\phi \in \mathcal{M}_b(\mathbf{R}^n)$ and that

$$(3.34) \quad \|\lambda_\phi\|_* \leq \|\phi\| \|\lambda\|_*.$$

This inequality is dual to the one that says that $\|\phi f\| \leq \|\phi\| \|f\|$ when $\phi, f \in C_b(\mathbf{R}^n)$.

Now suppose that $\lambda \in \mathcal{M}_b(\mathbf{R}^n)$ and that $\phi_1, \dots, \phi_l \in C_b(\mathbf{R}^n)$, and let us show that

$$(3.35) \quad \sum_{j=1}^l \|\lambda_{\phi_j}\|_* \leq \left\| \sum_{j=1}^l |\phi_j| \right\| \|\lambda\|_*.$$

This is equivalent to saying that if $f_1, \dots, f_l \in C_{00}(\mathbf{R}^n)$ and $\|f_j\| \leq 1$ for $j = 1, \dots, l$, then

$$(3.36) \quad \sum_{j=1}^l |\lambda(\phi_j f_j)| \leq \left\| \sum_{j=1}^l |\phi_j| \right\| \|\lambda\|_*.$$

This is equivalent in turn to saying that for such f_1, \dots, f_l and for complex numbers $\theta_1, \dots, \theta_l$ with $|\theta_j| = 1$ for $j = 1, \dots, l$ we have that

$$(3.37) \quad \left| \sum_{j=1}^l \theta_j \lambda(\phi_j f_j) \right| \leq \left\| \sum_{j=1}^l |\phi_j| \right\| \|\lambda\|_*.$$

Since λ is linear, this reduces to

$$(3.38) \quad \left| \lambda \left(\sum_{j=1}^n \theta_j \phi_j f_j \right) \right| \leq \left\| \sum_{j=1}^l |\phi_j| \right\| \|\lambda\|_*,$$

which holds because

$$(3.39) \quad \left\| \sum_{j=1}^l \theta_j \phi_j f_j \right\| \leq \left\| \sum_{j=1}^l |\phi_j| \right\|$$

under our conditions on the f_j 's and the θ_j 's.

Let λ be a bounded measure on \mathbf{R}^n , and let $\epsilon > 0$ be given. Let f be a continuous function on \mathbf{R}^n with compact support such that $\|f\| \leq 1$ and

$$(3.40) \quad |\lambda(f)| \geq \|\lambda\|_* - \epsilon.$$

Let ϕ be a real-valued continuous function on \mathbf{R}^n with compact support such that $0 \leq \phi(x) \leq 1$ for all $x \in \mathbf{R}^n$ and $\phi(x)f(x) = f(x)$ for all $x \in \mathbf{R}^n$. Thus

$$(3.41) \quad \lambda_\phi(f) = \lambda(f)$$

and therefore

$$(3.42) \quad \|\lambda_\phi\|_* \geq |\lambda_\phi(f)| \geq \|\lambda\|_* - \epsilon.$$

We also have that $\lambda_\phi + \lambda_{1-\phi} = \lambda$ and

$$(3.43) \quad \|\lambda\|_* = \|\lambda_\phi\|_* + \|\lambda_{1-\phi}\|_*$$

by the result of the preceding paragraph, and therefore

$$(3.44) \quad \|\lambda - \lambda_\phi\|_* = \|\lambda_{1-\phi}\|_* \leq \epsilon.$$

In other words, bounded measures on \mathbf{R}^n can be approximated by measures with compact support in the dual norm on measures.

Suppose that λ is a bounded measure on \mathbf{R}^n , and that $\{\lambda_j\}_{j=1}^\infty$ is a sequence of measures on \mathbf{R}^n with compact support which converge to λ in the dual norm, which is to say that

$$(3.45) \quad \lim_{j \rightarrow \infty} \|\lambda_j - \lambda\|_* = 0.$$

Let f be a bounded continuous function on \mathbf{R}^n . Thus $\lambda_j(f)$ is defined for all j , since the λ_j 's have compact support. Moreover,

$$(3.46) \quad |\lambda_j(f) - \lambda_l(f)| \leq \|\lambda_j - \lambda_l\|_* \|f\|$$

for all positive integers j, l , and it follows that $\{\lambda_j(f)\}_{j=1}^\infty$ is a Cauchy sequence of complex numbers. Every Cauchy sequence of complex numbers converges, and so we may define $\lambda(f)$ for $f \in C_b(\mathbf{R}^n)$ by

$$(3.47) \quad \lambda(f) = \lim_{j \rightarrow \infty} \lambda_j(f).$$

Of course this agrees with the initial definition of $\lambda(f)$ when $f \in C_{00}(\mathbf{R}^n)$.

If we think of $C_0(\mathbf{R}^n)$ as a metric space equipped with the supremum metric, then a bounded measure on \mathbf{R}^n is a linear functional on the dense subspace $C_{00}(\mathbf{R}^n)$ of $C_0(\mathbf{R}^n)$ which is continuous with respect to the supremum metric, and in fact uniformly continuous, because of linearity. By standard results about metric spaces there is a unique continuous extension of the linear functional to $C_0(\mathbf{R}^n)$, and the extension is linear and bounded with the same norm as the original. The extension described in the previous paragraph agrees with this one on $C_0(\mathbf{R}^n)$, and goes further.

More precisely, the extension of a bounded measure λ on \mathbf{R}^n to $C_b(\mathbf{R}^n)$ described above enjoys a stronger continuity property, which is continuity with respect to restricted convergence in $C_b(\mathbf{R}^n)$. This means that if $\{f_m\}_{m=1}^\infty$ is a sequence of bounded continuous functions on \mathbf{R}^n which are uniformly bounded on \mathbf{R}^n and which converge uniformly on compact subsets of \mathbf{R}^n to a function $f \in C_b(\mathbf{R}^n)$, then the extension of λ satisfies

$$(3.48) \quad \lim_{m \rightarrow \infty} \lambda(f_m) = \lambda(f).$$

One can check this using the approximation of λ in the dual norm by measures on \mathbf{R}^n with compact support. Of course for a measure with compact support one has continuity with respect to uniform convergence on compact subsets of \mathbf{R}^n .

Because $C_{00}(\mathbf{R}^n)$ is dense in $C_b(\mathbf{R}^n)$ with respect to this kind of restricted convergence, we have that our extension of λ to $C_b(\mathbf{R}^n)$ is uniquely determined by the initial definition of λ on $C_{00}(\mathbf{R}^n)$ and this continuity with respect to restricted convergence in $C_b(\mathbf{R}^n)$. In particular the extension of λ to $C_b(\mathbf{R}^n)$ does not depend on the choice of sequence of measures on \mathbf{R}^n with compact support which approximate λ , although this is easy to check directly too. Of course this extension of λ to $C_b(\mathbf{R}^n)$ is linear and satisfies

$$(3.49) \quad |\lambda(f)| \leq \|\lambda\|_* \|f\|$$

for all $f \in C_b(\mathbf{R}^n)$.

4 Fourier transforms

Let λ be a bounded measure on \mathbf{R}^n . For each $\xi \in \mathbf{R}^n$, let e_ξ denote the bounded continuous function on \mathbf{R}^n given by

$$(4.1) \quad e_\xi(x) = \exp(-2\pi i \xi \cdot x),$$

with

$$(4.2) \quad \xi \cdot x = \sum_{j=1}^n \xi_j x_j,$$

$x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$. As in the previous section, $\lambda(f)$ is defined when f is a bounded continuous function on \mathbf{R}^n , and we define the Fourier

transform of λ by

$$(4.3) \quad \widehat{\lambda}(\xi) = \lambda(e_\xi).$$

If $\xi \in \mathbf{C}^n$ we can define $e_\xi(x)$ as a continuous function on \mathbf{R}^n in exactly the same manner as above. If λ is a measure on \mathbf{R}^n with compact support, then $\lambda(h)$ is defined for all continuous functions on \mathbf{R}^n . In particular, $\widehat{\lambda}(\xi) = \lambda(e_\xi)$ is then defined for all $\xi \in \mathbf{C}^n$, and in fact it is a complex analytic function of ξ , which is to say that $\widehat{\lambda}(\xi)$ can be expressed as a power series in the ξ_j 's. In general for bounded measures λ it may be possible to define the Fourier transform of λ for some or all $\xi \in \mathbf{C}^n$, depending on the behavior of λ .

If λ is a bounded measure on \mathbf{R}^n , then we have that

$$(4.4) \quad |\widehat{\lambda}(\xi)| \leq \|\lambda\|_* \quad \text{for all } \xi \in \mathbf{R}^n.$$

Moreover, $\widehat{\lambda}(\xi)$ is a uniformly continuous function on \mathbf{R}^n . Indeed, if λ has compact support in \mathbf{R}^n , then its Fourier transform is quite a bit more regular than that. In general, a bounded measure can be approximated by measures with compact support in the dual norm, and this means that the Fourier transform of a bounded measure can be approximated in the supremum norm by the Fourier transforms of measures with compact support, so that the uniform continuity of the Fourier transform of a bounded measure follows from the regularity of the Fourier transforms of measures with compact support.

For each continuous function f on \mathbf{R}^n and each $v \in \mathbf{R}^n$ define $\tau_v(f)$ to be the continuous function on \mathbf{R}^n given by

$$(4.5) \quad \tau_v(f)(y) = f(v - y).$$

If λ is a measure on \mathbf{R}^n and f is a continuous function on \mathbf{R}^n , we would like to define the convolution of λ and f to be the function on \mathbf{R}^n expressed by the formula

$$(4.6) \quad (\lambda * f)(v) = \lambda(\tau_v(f)).$$

This makes sense if f is a continuous function on \mathbf{R}^n with compact support and λ is any measure on \mathbf{R}^n , in which case $\lambda * f$ is a continuous function on \mathbf{R}^n .

Alternatively, if λ is a measure on \mathbf{R}^n with compact support, and if f is any continuous function on \mathbf{R}^n , then $(\lambda * f)(v)$ is again defined. One can check that $\lambda * f$ is also continuous in this event, and indeed for a compact set of v 's $(\lambda * f)(v)$ will involve the values of f only on a compact subset of \mathbf{R}^n . If both λ and f have compact support, then $\lambda * f$ has compact support as well.

Now suppose that λ is a bounded measure on \mathbf{R}^n , and that f is a bounded continuous function on \mathbf{R}^n . In this case $\lambda * f$ can be defined again, and one can check that $\lambda * f$ is continuous using the fact that λ is continuous on $C_b(\mathbf{R}^n)$ with respect to restricted convergence of sequences of bounded continuous functions. Of course

$$(4.7) \quad \|\lambda * f\| \leq \|\lambda\|_* \|f\|.$$

One can also look at the continuity of $\lambda * f$ in terms of approximations of λ in the dual norm by measures with compact support, which lead to approximations of

$\lambda * f$ by convolutions of f with measures with compact support in the supremum metric on $C_b(\mathbf{R}^n)$.

If λ is a bounded measure on \mathbf{R}^n and $\xi \in \mathbf{R}^n$, then we have that

$$(4.8) \quad \lambda * e_\xi = \widehat{\lambda}(-\xi) e_\xi.$$

Thus the exponential function $e_x i$ is an eigenfunction for the linear operator

$$(4.9) \quad T_\lambda(f) = \lambda * f$$

on $C_b(\mathbf{R}^n)$, and the corresponding eigenvalue is given by the Fourier transform of λ at $-\xi$. If λ has compact support, then we can think of T_λ as a linear mapping from continuous functions on \mathbf{R}^n to themselves, and e_ξ is an eigenfunction of T_λ for all $\xi \in \mathbf{C}^n$, with eigenvalue equal to the Fourier transform of λ at $-\xi$.

References

- [1] S. Krantz, *A Panorama of Harmonic Analysis*, Carus Mathematical Monographs **27**, Mathematical Association of America, 1999.
- [2] E. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Lectures in Analysis **1**, Princeton University Press, 2003.
- [3] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series **32**, Princeton University Press, 1971.